ADVANCED PARTIAL DIFFERENTIAL EQUATIONS: HOMEWORK 3

KELLER VANDEBOGERT

1. Chapter 2, Problem 7

We have by the triangle inequality $|y|-|x| \leqslant |y-x| \leqslant |y|+|x|$. For $y \in \partial B(0,r)$, $|y| = r$, so that

$$\frac{1}{(r+|x|)^n} \leqslant \frac{1}{|y-x|^n} \leqslant \frac{1}{(r-|x|)^n}$$

On $\partial B(0,r)$. Suppose now that u is harmonic, and in particular satisfies the mean value property. Using the above and Poisson's formula for the ball:

$$\frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)} \frac{u(y)}{(r+|x|)^n} dS(y) \leqslant u(x) \leqslant \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)} \frac{u(y)}{(r-|x|)^n} dS(y)$$

Where we've used that $u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)} \frac{u(y)}{|y-x|^n}$ by Poisson's formula. Multiplying the left most and right most terms in the above inequality by r^{n-2}, we find:

$$\frac{(r-|x|)(r+|x|)r^{n-2}}{n\alpha(n)r^{n-1}(r+|x|)^n}\int_{\partial B(0,r)} u(y)dS(y) \leqslant u(x) \leqslant \frac{(r-|x|)(r+|x|)r^{n-2}}{n\alpha(n)r^{n-1}(r-|x|)^n}\int_{\partial B(0,r)} u(y)dS(y)$$

Which, by definition, becomes:

$$\frac{(r-|x|)r^{n-2}}{(r+|x|)^{n-1}} ⨍_{\partial B(0,r)} u(y)dS(y) \leqslant u(x) \leqslant \frac{(r+|x|)r^{n-2}}{(r-|x|)^{n-1}} ⨍_{\partial B(0,r)} u(y)dS(y)$$

Date: September 3, 2017.

But u is harmonic, so by the mean value property, $\fint_{\partial B(0,r)} u(y) = u(0)$. Using this, the above implies:

$$\frac{(r-|x|)r^{n-2}}{(r+|x|)^{n-1}}u(0) \leqslant u(x) \leqslant \frac{(r+|x|)r^{n-2}}{(r-|x|)^{n-1}}u(0)$$

As desired.

2. Chapter 2, Problem 8

Following this hint, we can see that since $u \equiv 1$ solves $\Delta u = 0$ in $B(0,r)$ and $u = 1$ on $\partial B(0,r)$, Poisson's formula for the ball implies:

$$1 = \int_{\partial B(0,r)} K(x,y)dy$$

Where $K(x,y)$ denotes Poisson's kernel. By definition, $G(x,y)$ for the ball is harmonic and smooth for $x \neq y$ and hence given $\epsilon > 0$, we can find δ such that whenever $|x - x_0| < \delta$, $|D^\alpha K(x,y) - D^\alpha K(x_0,y)| < \epsilon$ for any order derivative. In particular, since $\partial B(0,r)$ is compact we see that $D^\alpha u(x) = \int_{\partial B(0,r)} D^\alpha_x K(x,y)g(y)dy$. Using this, let $\epsilon > 0$, and note that since g is continuous on a compact set it is bounded:

(2.1)
$$\begin{aligned} |D^\alpha u(x) - D^\alpha u(x_0)| &\leqslant \int_{\partial B(0,r)} |D^\alpha K(x,y) - D^\alpha K(x_0,y)||g(y)|dy \\ &< \epsilon \int_{\partial B(0,r)} |g(y)|dy \\ &\leqslant \epsilon ||g||_{L^\infty} S_n \to 0 \end{aligned}$$

Where S_n denotes the surface area of the n-sphere. Hence u is smooth. Also, it is clear by the above that

$$\Delta u(x) = \int_{\partial B(0,r)} \Delta_x K(x,y)g(y)dy = 0$$

So that u is indeed harmonic. Finally, we must show that $u(x) = g(x)$ on $\partial B(0,r)$. Equivalently, this implies that $\lim_{x\to x_0} u(x) = g(x_0)$ when $x_0 \in \partial B(0,r)$. By continuity of g, we can find δ for any $\epsilon > 0$ such that $|g(x) - g(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. First, note that $\int_{\partial B(0,r)} g(x_0)K(x,y)dy = g(x_0)$. Then we can split up our integral as $\int_{\partial B(0,r)} = \int_{\partial B(0,r)\cap B(x_0,\delta)} + \int_{\partial B(0,r)\setminus B(x_0,\delta)}$. We have:

$$
\begin{aligned}
|u(x) - g(x_0)| &\leqslant \int_{\partial B(0,r)\cap B(x_0,\delta)} K(x,y)|g(y) - g(x_0)|dy \\
&\quad + \int_{\partial B(0,r)\setminus B(x_0,\delta)} K(x,y)|g(y) - g(x_0)|dy \\
&:= I + J
\end{aligned}
\tag{2.2}
$$

Then, firstly, since $1 = \int_{\partial B(0,r)} K(x,y)dy$, in particular $\int_U K(x,y) < 1$ for any $U \subset B(0,r)$, so we immediately find that

$$
I < \epsilon \int_{\partial B(0,r)\cap B(x_0,\delta)} K(x,y)dy < \epsilon
$$

For J, we see that by the triangle inequality $|y-x_0| \leqslant |y-x|+|x-x_0|$ and $|g(x) - g(x_0) \leqslant 2||g||_{L^\infty}$. In particular, it is clear that for $y \notin B(x_0,\delta)$ and $|x - x_0| < \delta/2$, that $|y - x_0| \leqslant |y - x| + \frac{1}{2}|y - x_0| \implies \frac{1}{2}|y - x_0| \leqslant |y - x|$. Employing this in J:

$$
\begin{aligned}
\int_{\partial B(0,r)\setminus B(x_0,\delta)} K(x,y)|g(y) - g(x_0)|dy &\leqslant \frac{2(r^2 - |x|^2)||g||_{L^\infty}}{n\alpha(n)r} \int_{\partial B(0,r)\setminus B(x_0,\delta)} |y - x_0|^{-n}dy \\
&\to 0
\end{aligned}
\tag{2.3}
$$

Since $|x| \to r$ as $x \to x_0$, as $x_0 \in \partial B(0,r)$. Then we are done, since we see that $|u(x) - g(x_0)| \to 0$ as $x \to x_0$, so that this is indeed a solution.

3. Chapter 2, Problem 10

(a). Denoting by v for the odd reflection of $u(x_1, \dots, x_n)$, when $x_n > 0$, $u \equiv v$ so v is obviously C^2 on the upper half. For $x_n < 0$:

$$\frac{\partial v}{\partial x_i} = -\frac{\partial u}{\partial x_i} \quad i \neq n$$
$$\frac{\partial v}{\partial x_n} = \frac{\partial u}{\partial x_n}$$

Which are both continuous in the lower half sphere. Similarly,

$$\frac{\partial^2 v}{\partial x_i^2} = -\frac{\partial^2 u}{\partial x_i^2} \quad i \neq n$$
$$\frac{\partial^2 v}{\partial x_n^2} = -\frac{\partial^2 u}{\partial x_n^2}$$

Thus these derivatives are again continuous, since $u \in C^2(\overline{C^2})$. Using this, in the lower half sphere, we see that $\Delta v = -\Delta u = 0$, so v is harmonic in both the upper and lower half ball, and hence everywhere. Also, we see that on the intersection of their respective boundaries, $v = u(x_1, \dots, x_{n-1}, 0) = 0$.

(b). We can employ Poisson's formula for the ball to find that

$$v(x) = \int_{\partial B(0,1)} K(x,y) v(y) dS(y)$$

Where $K(x,y)$ denotes Poisson's kernel. We have shown that Poisson's kernel is harmonic and that $K(x,y) \in C^2(\overline{B(0,1)})$, and so we find:

$$\frac{\partial v(x)}{\partial x_i} = \int_{\partial B(0,1)} \frac{\partial K(x,y)}{\partial x_i} v(y) dS(y)$$

And since we know that $v(y)$ is continuous on $\partial B(0,1)$, we see that the product $\frac{\partial K(x,y)}{\partial x_i}v(y)$ is continuous and hence so is the integral, so that $v(x) \in C^1(\overline{U})$. Similarly, the second derivative is easy to calculate:

$$\frac{\partial^2 v(x)}{\partial x_i^2} = \int_{\partial B(0,1)} \frac{\partial^2 K(x,y)}{\partial x_i^2} v(y) dS(y)$$

So that $v \in C^2(\overline{U})$. Finally, noting that $K(x,y)$ is harmonic:

$$\Delta v(x) = \int_{\partial B(0,1)} \Delta_x K(x,y) v(y) dS(y) = 0$$

So that v is harmonic, and we are done.

4. Chapter 2, Problem 12

(a). Define $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$. Then, we see:

$$\frac{\partial u_\lambda}{\partial t} = \lambda^2 \frac{\partial u}{\partial t}(\lambda x, \lambda^2 t)$$

$$\frac{\partial^2 u_\lambda}{\partial x_i^2} = \lambda^2 \frac{\partial^2 u}{\partial x_i^2}(\lambda x, \lambda^2 t)$$

And hence summing over all i:

$$\frac{\partial u_\lambda}{\partial t} - \Delta u_\lambda = \lambda^2 (u_t - \Delta u) = 0$$

So that u_λ also satisfies the heat equation.

(b). Take the partial with respect to λ:

$$\frac{\partial u_\lambda}{\partial \lambda} = x \cdot Du(\lambda x, \lambda^2 x) + 2\lambda t \frac{\partial u}{\partial t}(\lambda x, \lambda^2 x)$$

However, by the above it is clear that $v(x,t) = \frac{\partial u_\lambda}{\partial t}|_{\lambda=0}$, v as defined in the book. Using commutativity of mixed partial derivatives (guaranteed by the smoothness of u):

$$\begin{aligned} v_t - \Delta v &= \frac{\partial^2}{\partial t \partial \lambda} u_\lambda - \Delta \frac{\partial}{\partial \lambda} u_\lambda \\ &= \frac{\partial}{\partial \lambda}\Big(\frac{\partial u_\lambda}{\partial t} - \Delta u_\lambda \Big) = 0 \end{aligned} \tag{4.1}$$

Since $\frac{\partial u_\lambda}{\partial t} - \Delta u_\lambda = 0$ by part (a). Hence we see that after setting $\lambda = 1$, v satisfies the heat equation as well.

5. Chapter 2, Problem 14

Define $v := e^{ct}u$. Then, by the statement of the problem, we see that v satisfies

$$v_t - \Delta u = e^{ct} f$$

$$v = g$$

In $\mathbb{R}^{n+1}_+$ and on $\partial \mathbb{R}^{n+1}_+$, respectively. Letting $\Phi(x,t) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$ denote the heat kernel, we can solve for v:

$$v(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) dy ds$$

And using the fact that $v(x,t) := e^{ct}u(x,t)$, multiply the above by e^{-ct} to find:

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) e^{-ct} g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) e^{-ct} f(y,s) dy ds$$

And the above solves $u_t - \Delta u + cu = f$ in $\mathbb{R}^{n+1}_+$ and $u = g$ on $\partial \mathbb{R}^{n+1}_+$, so we are done.

6. Chapter 2, Problem 15

Define $v(x,t) := u(x,t) - g(t)$. Then, it is clear that v satisfies $v_t - v_{xx} = -g'$ in $\mathbb{R}_+ \times (0,\infty)$, $v(x,0) = g(0) = 0$, and $v(0,t) = 0$. We now extend v by odd reflection, so that for $x < 0$, $v(x,t) := -v(-x,t)$. Then, this extends v to satisfy $v_t - v_{xx} = -g'(t)$ when $x \geqslant 0$, $v_t - v_{xx} - = g'(t)$ for $x < 0$, and $v(x,0) = 0$. Then, using Duhamel's principle we can immediately solve for v:

$$v(x,t) = \int_0^t \frac{1}{(4\pi(t-s))^{1/2}} \Big(-\int_0^\infty e^{\frac{-(x-y)^2}{4(t-s)}} g'(s)dy + \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} g'(s)dy \Big) ds$$

Now, let us consider the term $\int_0^\infty e^{\frac{-(x-y)^2}{4(t-s)}} g'(s)dy$. We have:

$$\begin{aligned}
\int_0^\infty e^{\frac{-(x-y)^2}{4(t-s)}} g'(s)dy &= \int_{-\infty}^\infty e^{\frac{-(x-y)^2}{4(t-s)}} g'(s)dy - \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} g'(s)dy \\
&= \int_{-\infty}^\infty e^{\frac{-x^2}{4(t-s)}} g'(s)dy - \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} g'(s)dy \\
&= g'(s)\big(4(t-s)\big)^{1/2}\Gamma(1/2) - \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} g'(s)dy
\end{aligned} \tag{6.1}$$

Plugging this back into our expression for v (and noting that $\Gamma(1/2) = \sqrt{\pi}$):

$$v(x,t) = -\int_0^t g'(s)ds + 2\int_0^t \frac{1}{(4\pi(t-s))^{1/2}} \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} g'(s)dyds$$

Now, of course $\int_0^t g'(s) = g(t)$, so using the fact that $v(x,t) = u(x,t) - g(t)$, we have an expression for $u(x,t)$:

$$u(x,t) = 2\int_0^t \frac{1}{(4\pi(t-s))^{1/2}} \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} g'(s)dyds$$

We must now integrate by parts:

$$\int_0^t \frac{1}{(4\pi(t-s))^{1/2}} \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} dy dg(s) = \frac{g(s)}{(4\pi(t-s))^{1/2}} \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} dy \Big|_0^t - \int_0^t \frac{g(s)}{4\sqrt{\pi}(t-s)^{3/2}} \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} dy ds + \int_0^t \frac{g(s)}{4\sqrt{\pi}(t-s)^{3/2}} \int_{-\infty}^0 \frac{(x-y)^2 e^{\frac{-(x-y)^2}{4(t-s)}}}{2(t-s)} dy ds := I - J + K \tag{6.2}$$

We can integrate the term $\int_{-\infty}^0 \frac{(x-y)^2 e^{\frac{-(x-y)^2}{4(t-s)}}}{2(t-s)} dy$ by noticing that $\frac{d}{dy} e^{\frac{-(x-y)^2}{4(t-s)}} = \frac{(x-y)e^{\frac{-(x-y)^2}{4(t-s)}}}{2(t-s)}$. Then:

$$\int_{-\infty}^0 (x-y) d\Big(e^{\frac{-(x-y)^2}{4(t-s)}}\Big) = \Big[(x-y)e^{\frac{-(x-y)^2}{4(t-s)}}\Big]_{-\infty}^0 + \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} dy = xe^{\frac{-x^2}{4(t-s)}} + \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} dy \tag{6.3}$$

Plugging the above into the last term, K, of (6.2):

$$\int_0^t \frac{g(s)}{4\sqrt{\pi}(t-s)^{3/2}} \int_{-\infty}^0 \frac{(x-y)^2 e^{\frac{-(x-y)^2}{4(t-s)}}}{2(t-s)} dy ds = x\int_0^t \frac{xe^{\frac{-x^2}{4(t-s)}} g(s)}{4\sqrt{\pi}(t-s)^{3/2}} ds + \int_0^t \frac{g(s)}{4\sqrt{\pi}(t-s)^{3/2}} \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} dy ds \tag{6.4}$$

But this shows that $K = x\int_0^t \frac{xe^{\frac{-x^2}{4(t-s)}} g(s)}{4\sqrt{\pi}(t-s)^{3/2}} + J$, and hence, using this in (6.2):

(6.5)

$$\int_0^t \frac{1}{(4\pi(t-s))^{1/2}} \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4(t-s)}} g'(s)dyds = I + x\int_0^t \frac{xe^{\frac{-x^2}{4(t-s)}} g(s)}{4\sqrt{\pi}(t-s)^{3/2}}ds$$

We proceed to show that $I = 0$. Since $g(0) = 0$, it is obvious that the term inside I evaluated at 0 is 0, since all other terms remain finite. To find the inside of I evaluated at t, suppose that $t - s < \epsilon$ and then let $\epsilon \to 0^+$. Then, it is clear that $\frac{e^{\frac{-(x-y)^2}{4\epsilon}}}{\epsilon^{1/2}} \to 0$ uniformly as $\epsilon \to 0^+$, $x \neq y$, and hence evaluating at t also yields 0, since the only point for which it does not tend to 0 is a singleton and hence integrates to 0. Thus, $I = 0$ and combining all of the above with this, we can finally conclude, using our expression for $u(x,t)$:

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{e^{\frac{-x^2}{4(t-s)}} g(s)}{(t-s)^{3/2}}ds$$

7. Chapter 2, Problem 16

Defining $u_\epsilon := u - \epsilon t$ ($\epsilon > 0$), we see that $u_{\epsilon t} - \Delta u_\epsilon = -\epsilon < 0$. This shows that u_ϵ cannot attain a maximum on the interior of U_T, since if this were the case, we would see that $u_{\epsilon t} \geqslant 0$ (since locally, u must be increasing) and $\Delta u_\epsilon \leqslant 0$ (locally convex) for some $(x_0, t_0) \in U_T$. But this would then force $u_{\epsilon t} - \Delta u_\epsilon \geqslant 0$. Hence, u_ϵ cannot attain is maximum on the interior.

Now, denote by M_ϵ the maximum of u_ϵ on the boundary Γ_T, and $M := \max_{\overline{U}_T} u$. By our definition, we certainly have that $u_\epsilon \leqslant u$. Suppose now for sake of contradiction that u attains its maximum on the interior of U_T. Then, $M - \epsilon t \leqslant M_\epsilon$, since else u_ϵ would attain its maximum on the interior as well. Hence we have the following chain of inequalities:

$$M - \epsilon t \leqslant M_\epsilon \leqslant M$$

Letting $\epsilon \to 0$, we see that $M_0 = M$. But $M_0 = \max_{\Gamma_T} u$, hence we conclude:

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u$$

Contradicting the fact that u attains its maximum on the interior. Hence the assertion follows.

8. Chapter 2, Problem 17

(a). Let $E(r) := E(0,0;r)$, where u is a subsolution of the heat equation. Without loss of generality, we can assume $(x,t) = (0,0)$. Then, define $\psi(y,s) := -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log(r)$, where $s \leqslant 0$, and note that ψ restricted to $\partial E(r)\backslash(0,0)$ vanishes. Define:

$$\begin{aligned} \phi(r) &:= \frac{1}{r^n}\iint_{E(r)} u(y,s)\frac{|y|^2}{s^2}dyds \\ &= \iint_{E(1)} u(ry, r^2 s)\frac{|y|^2}{s^2}dyds \end{aligned} \tag{8.1}$$

The second equality comes from the natural change of variable $y \mapsto ry$, $s \mapsto r^2 s$. Now we can differentiate ϕ with respect to r:

$$\begin{aligned} \phi'(r) &= \iint_{E(1)} Du \cdot y\frac{|y|^2}{s^2} + 2ru_s\frac{|y|^2}{s^2}dyds \\ &= \frac{1}{r^{n+1}}\iint_{E(r)} Du \cdot y\frac{|y|^2}{s^2} + 2u_s\frac{|y|^2}{s^2}dyds \\ &:= I + J \end{aligned} \tag{8.2}$$

Now we want to consider the term J. Note that $|y|^2/s^2 = 2D\psi \cdot y$, where ψ defined above. Using this,

$$J = \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s D\psi \cdot y dyds$$

Now integrate by parts with respect to y, and note that our boundary term vanishes. Then, $D(u_s y) = Du_s \cdot y + Dy \cdot u_s$. But Dy is merely a vector of 1's, and hence $Dy \cdot u = nu_s$. Using this,

(8.3)
$$\frac{1}{r^{n+1}} \iint_{E(r)} 4u_s D\psi \cdot y dyds = -\frac{1}{r^{n+1}} \iint_{E(r)} 4\psi Du_s \cdot y + 4nu_s\psi dyds$$

Now integrate the first term by parts with respect to s. Then, $\psi_s = -\frac{n}{2s} - \frac{|y|^2}{4s^2}$. Using this:

(8.4)
$$\begin{aligned} -\frac{1}{r^{n+1}} \iint_{E(r)} 4\psi Du_s \cdot y + 4nu_s\psi dyds &= -\frac{1}{r^{n+1}} \iint_{E(r)} -4\psi_s Du \cdot y + 4nu_s\psi dyds \\ &= -\frac{1}{r^{n+1}} \iint_{E(r)} \frac{2n}{s} Du \cdot y + \frac{|y|^2}{s^2} Du \cdot y + 4nu_s\psi dyds \\ &= -\frac{1}{r^{n+1}} \iint_{E(r)} \frac{2n}{s} Du \cdot y + 4nu_s\psi dyds - I \end{aligned}$$

Combining this with our expression for $\phi'(r)$:

$$\phi'(r) = -\frac{1}{r^{n+1}} \iint_{E(r)} \frac{2n}{s} Du \cdot y + 4nu_s\psi dyds$$

But u is a subsolution. Hence, $u_s \leqslant \Delta u$. We want to integrate this by parts:

(8.5)
$$\begin{aligned} -\frac{1}{r^{n+1}} \iint_{E(r)} \frac{2n}{s} Du \cdot y + 4nu_s\psi dyds &\geqslant -\frac{1}{r^{n+1}} \iint_{E(r)} \frac{2n}{s} Du \cdot y + 4n\Delta u\psi dyds \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} -\frac{2n}{s} Du \cdot y + 4nDu \cdot D\psi dyds \end{aligned}$$

Noting the form of ψ, we see that $D\psi = \frac{y}{2s}$. Hence:

$$\frac{1}{r^{n+1}}\iint_{E(r)} -\frac{2n}{s}Du\cdot y+4nDu\cdot D\psi dyds = \frac{1}{r^{n+1}}\iint_{E(r)} Du\cdot\big(-\frac{2n}{s}y+4n\frac{y}{2s}\big)dyds = 0$$

Combining this with the above, we find that $\phi'(r) \geqslant 0$, so that ϕ is an increasing function. This means that it attains its minimum as $r \to 0$.

$$\begin{aligned}\lim_{r\to 0}\phi(r) &= \lim_{r\to 0}\frac{1}{r^n}\iint_{E(r)} u(y,s)\frac{|y|^2}{s^2}dyds \\ &= \lim_{r\to 0}\iint_{E(1)} u(ry,r^2s)\frac{|y|^2}{s^2}dyds \\ &= u(0,0)\iint_{E(1)}\frac{|y|^2}{s^2}dyds\end{aligned} \tag{8.6}$$

Our task is to compute $\iint_{E(1)}\frac{|y|^2}{s^2}dyds$. Recall that

$$E(1) = \{(y,s):\ s\leqslant 0,\ \frac{e^{-\frac{|y|^2}{4s}}}{(-4\pi s)^{n/2}}\geqslant 1\}$$

Then, by basic manipulations we see that $|y|^2 \leqslant 2ns\log(-4\pi s)$ in $E(1)$. Also, since $|y|^2 \geqslant 0$, we see that $s \geqslant -\frac{1}{4\pi}$. Hence,

$$\iint_{E(1)}\frac{|y|^2}{s^2}dyds = \int_{-1/4\pi}^{0}\frac{1}{s^2}\int_{|y|^2\leqslant 2ns\log(-4\pi s)}|y|^2dyds$$

Converting to spherical coordinates, $|y|\mapsto r$:

(8.7)

$$\begin{aligned}\int_{-1/4\pi}^{0}\frac{1}{s^2}\int_{|y|^2\leqslant 2ns\log(-4\pi s)}|y|^2dyds &= \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_{-1/4\pi}^{0}\frac{1}{s^2}\int_0^{(2ns\log(-4\pi s))^{1/2}} r^{n+1}drds \\ &= \frac{2\pi^{n/2}}{(n+2)\Gamma(n/2)}\int_{-1/4\pi}^{0}(2ns\log(-4\pi s))^{n/2+1}s^{-2}ds \\ &= \frac{2\cdot(2n)^{n/2+1}\pi^{n/2}}{(n+2)\Gamma(n/2)}\int_0^{1/4\pi} s^{n/2-1}\Big(\log\big(\frac{1}{4\pi s}\big)\Big)^{n/2+1}ds\end{aligned}$$

Now make the change of variable $u = \log\left(\frac{1}{4\pi s}\right)$. Our bounds will change as $\int_0^{1/4\pi} \to \int_\infty^0$:

$$
\begin{aligned}
&\frac{2\cdot(2n)^{n/2+1}\pi^{n/2}}{(n+2)\Gamma(n/2)}\int_0^{1/4\pi} s^{n/2-1}\Big(\log\big(\frac{1}{4\pi s}\big)\Big)^{n/2+1} ds \\
&= \frac{2\cdot(2n)^{n/2+1}\pi^{n/2}}{(n+2)\Gamma(n/2)(4\pi)^{n/2}}\int_0^\infty (e^{-nu/2})u^{n/2+1}du \\
&= \frac{2\cdot(2n)^{n/2+1}\pi^{n/2}}{(n+2)\Gamma(n/2)(4\pi)^{n/2}}\Big(\frac{2}{n}\Big)^{n/2+2}\int_0^\infty e^{-t}t^{n/2+1}dt \\
&= \frac{2\cdot(2n)^{n/2+1}\pi^{n/2}}{(n+2)\Gamma(n/2)(4\pi)^{n/2}}\Big(\frac{2}{n}\Big)^{n/2+2}\Gamma(n/2+2) \\
&= \frac{(2n)^{n/2+1}}{(n/2+1)\Gamma(n/2)4^{n/2}}\Big(\frac{2}{n}\Big)^{n/2+2}(n/2+1)\Gamma(n/2+1) \\
&= \frac{(2n)^{n/2+1}}{\Gamma(n/2)2^n}\Big(\frac{2}{n}\Big)^{n/2+2}(n/2)\Gamma(n/2) \\
&= \frac{2^{n/2+1}n^{n/2+1}}{2^n}\frac{2^{n/2+2}}{n^{n/2+2}}(n/2) \\
&= \frac{2^{n+2}}{2^n} = 2^2 = 4
\end{aligned}
\tag{8.8}
$$

And hence we see $\lim_{r\to 0}\phi(r) = 4u(0,0)$. Since $(0,0)$ can be translated to any general coordinate (x,t), we conclude:

$$u(x,t) \leqslant \frac{1}{4r^n}\iint_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}dyds$$

As asserted, and we are done.

(b). Suppose u attains its maximum at the point (x_0,t_0). Then, by part a, whenever $0 < r < \operatorname{dist}((x,t),\partial U_T)$:

$$M = u(x_0,t_0) \leqslant \frac{1}{4r^n}\iint_{E(x_0,t_0;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}dyds \leqslant M\frac{1}{4r^n}\iint_{E(r)}\frac{|x-y|^2}{(t-s)^2}dyds$$

By the work of part (a), it is seen that $\frac{1}{4r^n}\iint_{E(x_0,t_0;r)}\frac{|x-y|^2}{(t-s)^2}dyds = 1$ by a simple change of variable. We then conclude that $u(x,t) = M$

for all $(x,t) \in E(x_0, t_0; r)$. However, the set $\{(x,t) : u(x,t) = M\}$ is a closed set and the above shows that it is also an open set. Since U_T is connected, the only clopen subset of U_T is the entire set itself. Then, $u(x,t) = M$ for all (x,t), implying u is constant.

Then, if we assumed that u were not constant and attained its maximum on the interior, we can apply the above argument to see that u attains its maximum in some heat ball contained in U_T. But this forces u to be constant, a contradiction. Hence, u attains its maximum on the boundary, and:

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u$$

As asserted.

(c). Define $v := \phi(u)$, where ϕ is smooth and convex, implying $\phi'' \geqslant 0$. Then, we compute:

$$\frac{\partial}{\partial x_i}\phi(u) = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x_i}$$

$$\frac{\partial}{\partial x_i}\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x_i} = \frac{\partial^2 \phi}{\partial u^2}\Big(\frac{\partial u}{\partial x_i}\Big)^2 + \frac{\partial \phi}{\partial u}\frac{\partial^2 u}{\partial x_i^2}$$

Summing over i, $\Delta v = \sum_i \frac{\partial^2 \phi}{\partial u^2}\left(\frac{\partial u}{\partial x_i}\right)^2 + \frac{\partial \phi}{\partial u}\Delta u$. Also, $v_t = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial t}$. Putting this all together:

$$\begin{aligned} v_t - \Delta v &= \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial t} - \sum_i \frac{\partial^2 \phi}{\partial u^2}\Big(\frac{\partial u}{\partial x_i}\Big)^2 - \frac{\partial \phi}{\partial u}\Delta u \\ &= \frac{\partial \phi}{\partial u}\Big(u_t - \Delta u\Big) - \sum_i \frac{\partial^2 \phi}{\partial u^2}\Big(\frac{\partial u}{\partial x_i}\Big)^2 \\ &= -\sum_i \frac{\partial^2 \phi}{\partial u^2}\Big(\frac{\partial u}{\partial x_i}\Big)^2 \leqslant 0 \end{aligned} \tag{8.9}$$

Where the last step uses convexity of ϕ. Then we see that this is indeed a subsolution.

(d). Define $v := |Du|^2 + u_t^2$, where u_t solves the heat equation. Then, we first calculate:

$$\frac{\partial}{\partial x_i}|Du|^2 = 2Du \cdot Du_{x_i}$$

$$\frac{\partial}{\partial x_i} 2Du \cdot Du_{x_i} = 2|Du_{x_i}|^2 + 2Du \cdot Du_{x_i x_i}$$

And hence, from the above, $\Delta|Du|^2 = 2\sum_i |Du_{x_i x_i}|^2 + 2Du \cdot D(\Delta u)$. Likewise, it is obvious that $\frac{\partial}{\partial t}|Du|^2 = 2Du \cdot Du_t$. We also see:

$$\frac{\partial}{\partial x_i} u_t^2 = 2u_t u_{tx_i}$$

$$\frac{\partial}{\partial x_i} 2u_t u_{tx_i} = 2u_{tx_i}^2 + 2u_t u_t x_i x_i$$

And similarly, summing over i, $\Delta u_t^2 = 2\sum_i u_{tx_i}^2 + 2u_t(\Delta u)_t$. Also, $\frac{\partial}{\partial t} u_t^2 = 2u_t u_{tt}$. Using all of the above:

$$\begin{aligned} v_t - \Delta v &= 2Du \cdot Du_t + 2u_t u_{tt} - 2\sum_i |Du_{x_i x_i}|^2 - 2Du \cdot D(\Delta u) - 2\sum_i u_{tx_i}^2 - 2u_t(\Delta u)_t \\ &= 2Du \cdot D(u_t - \Delta u) + 2u_t(u_t - \Delta u)_t - 2\sum_i \left(|Du_{x_i x_i}|^2 + u_{tx_i}^2\right) \\ &= -2\sum_i \left(|Du_{x_i x_i}|^2 + u_{tx_i}^2\right) \leqslant 0 \end{aligned}$$

So we conclude that v is indeed a subsolution as asserted.

9. Chapter 2, Problem 18

Suppose that $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$, with $u(x, 0) = 0$ and $u_t(x, 0) = h$. Then, set $v := u_t$.

It is clear that $v_{tt} - \Delta v = (u_{tt} - \Delta u)_t = 0$ in $\mathbb{R}^n \times (0, \infty)$. Similarly, $v(x, 0) = u_t(x, 0) = h$.

Finally, $v_t(x, 0) = u_{tt}(x, 0) = \Delta u(x, 0)$. But $u(x, 0) = 0$, and hence $\Delta u(x, 0) = 0$, so that $v_t(x, 0) = 0$, as desired.

10. Chapter 2, Problem 19

(a). Suppose $u_{xy} = 0$. Then, certainly $u_x = f(x)$, where f is any function. Integrating again, we see that $u(x, y) = \int_{x_0}^{x} f(x)dx + g(y)$. Redefining our functions, we conclude that $u(x, y) = F(x) + G(y)$ for arbitrary functions F and G.

(b). Set $\xi = x + t$ and $\eta = x - t$. Then, for $u(\xi, \eta)$, employ the chain rule to find:

$$u_t(\xi, \eta) = u_\xi - u_\eta$$
$$u_{tt}(\xi, \eta) = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

Similarly,

$$u_x(\xi, \eta) = u_\xi + u_\eta$$
$$u_{xx}(\xi, \eta) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Assuming first that $u_{tt} - u_{xx} = 0$, subtracting the above becomes $-4u_{\xi\eta} = 0$, so $u_{\xi\eta} = 0$. Conversely, if $u_{\xi\eta} = 0$, then, noting that $x = \eta/2 + \xi/2$ and $t = \xi/2 - \eta/2$:

$$u_\xi = u_x/2 + u_t/2$$
$$u_{\xi\eta} = u_{xx}/4 - u_{tt}/4$$

And hence $u_{tt} - u_{xx} = 0$.

(c). Using the previous two parts, we can immediately deduce that if $u_{tt} - u_{xx} = 0$, then $u = F(x+t) + G(x-t)$ for some functions F, G. Suppose that $u(x,0) = g$, $u_t(x,0) = h$. Then, this implies that $g(x) = F(x) + G(x)$ and $h(x) = F'(x) - G'(x)$. Solving for F and G, we find:

$$\begin{aligned} F'(x) &= \frac{1}{2}\big(g'(x) + h(x)\big) \\ G'(x) &= \frac{1}{2}\big(g'(x) - h(x)\big) \end{aligned}$$

And hence, after integrating we find:

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(\int_0^{x+t} g'(y) + h(y)dy + \int_0^{x-t} g'(y) - h(y)dy\Big) \\ &= \frac{1}{2}\Big(g(x+t) + g(x-t)\Big) + \frac{1}{2}\Big(\int_0^{x+t} h(y)dy - \int_0^{x-t} h(y)dy\Big) \\ &= \frac{1}{2}\Big(g(x+t) + g(x-t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \end{aligned} \tag{10.1}$$

Which is precisely D'Alembert's solution, so we are done.

(d). Suppose first that u is a left moving wave. In particular, this means that $u_x + u_t = 0$, where $u(x,0) = g(x)$. Then, this is merely the solution to the transport equation, so $u(x,t) = g(x-t)$. Using this, we also know that $u_t(x,0) = h(x)$. But $u_t = -g'(x-t) \implies u_t(x,0) = -g'(x)$. Then, we see that $h(x) = -g'(x)$.

Similarly, if we have that u is a right moving wave, u satisfies $u_t - u_x = 0$, with $u(x,0) = g(x)$. Again, using the solution to the transport equation, $u(x,t) = g(x+t)$. This then shows that $u_t = g'(x+t)$ and hence $g'(x) = h(x)$, and these are the conditions for the solution being a left and right moving wave.

11. Chapter 2, Problem 24

(a). Suppose u solves the wave equation in $\mathbb{R} \times (0, \infty)$. Then, with $k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$ and $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$. Then, u can of course be solved using D'Alembert's solution. Firstly, however, note:

$$\frac{d}{dt}k(t) = \int_{-\infty}^{\infty} u_t u_{tt} dx$$
$$\frac{d}{dt}p(t) = \int_{-\infty}^{\infty} u_x u_{xt} dx$$

Recalling that $u_{tt} = u_{xx}$, we can now integrate the expression $\int_{-\infty}^{\infty} u_t u_{tt} dx = \int_{-\infty}^{\infty} u_t u_{xx} dx$ by parts.

$$\int_{-\infty}^{\infty} u_t u_{xx} dx = u_t u_x \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_x u_{xt} dx$$

Since g and h have compact support and noticing the form of D'Alembert's solution, this immediately implies that $u_t u_x \Big|_{-\infty}^{\infty} = 0$. Then,

$$\frac{d}{dt}k(t) + \frac{d}{dt}p(t) = -\int_{-\infty}^{\infty} u_x u_{xt} dx + \int_{-\infty}^{\infty} u_x u_{xt} dx = 0$$

And hence $k(t) + p(t)$ is a constant with respect to t.

(b). Using parts (a) and (b) of the previous problem, we know that $u(x,t) = F(x+t) + G(x-t)$ for some functions F, G. Noting the dependence on our initial conditions for D'Alembert's solution, we can also conclude immediately that F and G must have compact support. Now, it is simple to see:

$$u_t = F'(x+t) - G'(x-t)$$

$$u_x = F'(x+t) + G'(x-t)$$

And hence:

$$\int_{-\infty}^{\infty} u_t^2 - u_x^2 dx = -4 \int_{-\infty}^{\infty} F'(x+t)G'(x-t)dx$$

But for any compactly supported function, its derivative must also be compactly supported. Suppose that the support of both of these is contained in the interval $[-M, M]$ for sufficiently large M. Then for any x, it is never possible for $x - 2M$ *and* $x + 2M$ to both be contained in $[-M, M]$, since else the length of this interval would be at least $4M$. In other words, for $t > 2M$, either $G'(x-t)$ or $F'(x+t)$ must vanish, implying that

$$k(t) = p(t)$$

For sufficiently large t.